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Discontinuous Galerkin methods for problems in fluid mechanics and aeroacoustics

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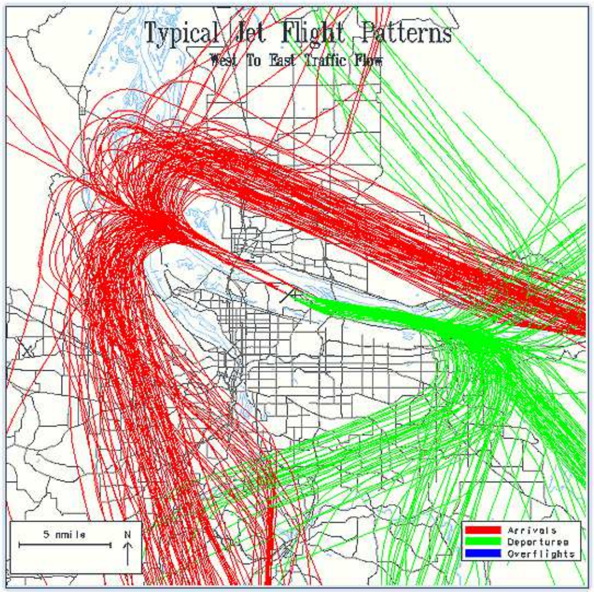
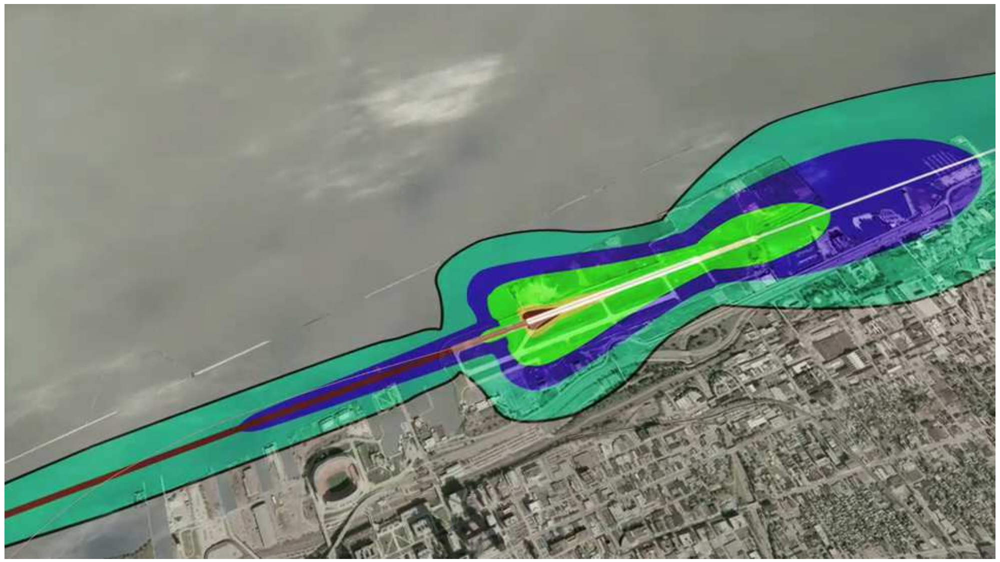
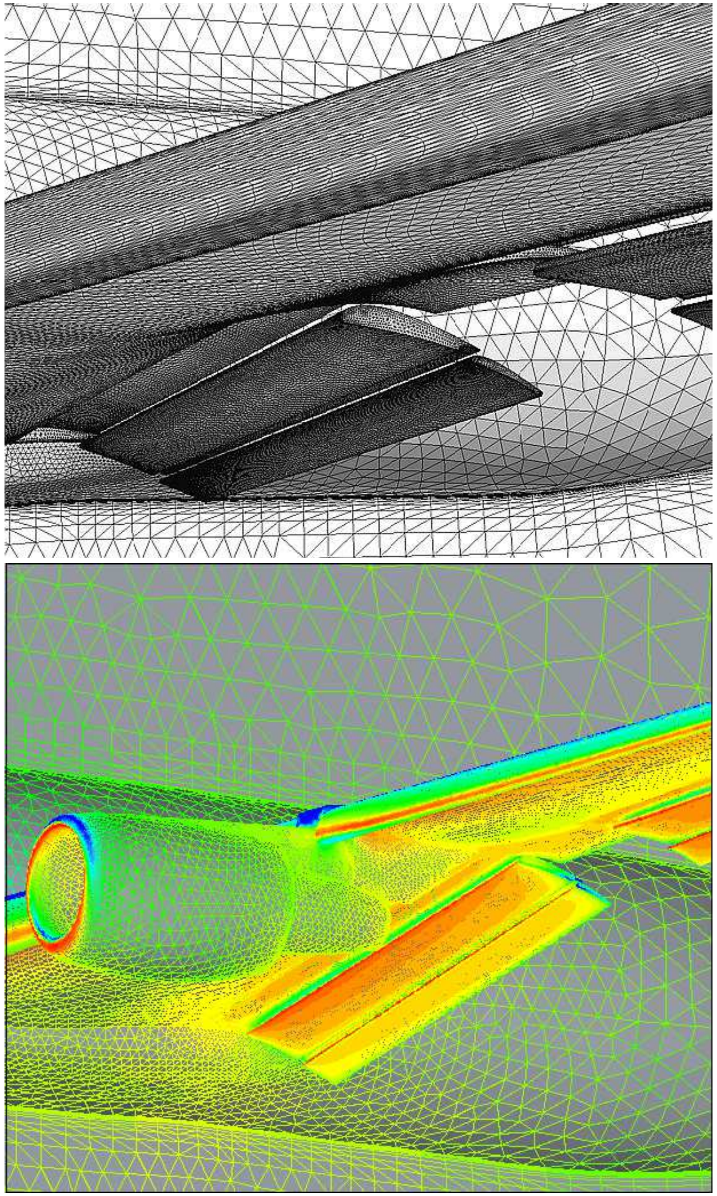
Noise generation is a major driver for aircraft development



- Noise generation biggest challenge after fuel efficiency
- Interaction between noise sources and aircraft structure
- Holistic approach to analysis necessary



Simulation of full aircraft is prohibitively expensive using standard methods

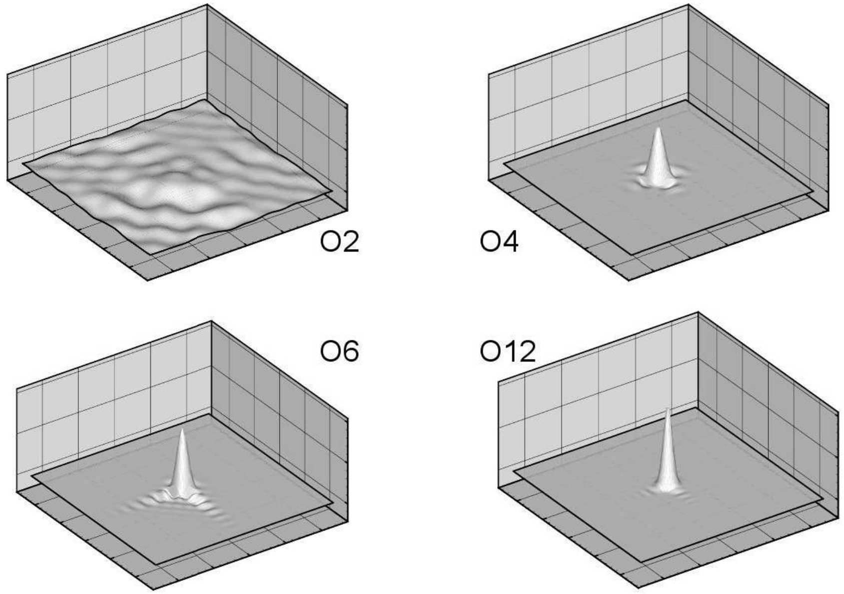


- With typical $\mathcal{O}(2)$ central-difference scheme, $\Omega = (200\mathrm{m})^3$, $f = 3000\mathrm{Hz}$: $\approx \mathbf{2.2\,10^{13}}$ grid points
- Fewer points necessary for higher order schemes
  - $\mathcal{O}(2)$ (CD scheme): 16 ppw
  - $\mathcal{O}(4)$ (DRP scheme): 6 ppw
  - $\mathcal{O}(6)$ (SBP scheme): 4 ppw
- Complex geometries require flexible mesh $\rightarrow$ structured grids ill-suited
- Problem size makes use of high performance computing mandatory

Scheme requirements and alternatives

A suitable method for CAA should...

- be conservative
- be useable on complex geometries
- support adaptive refinement
- be fast and allow efficient parallelization
- allow high order discretization
- have low dissipation & dispersion errors

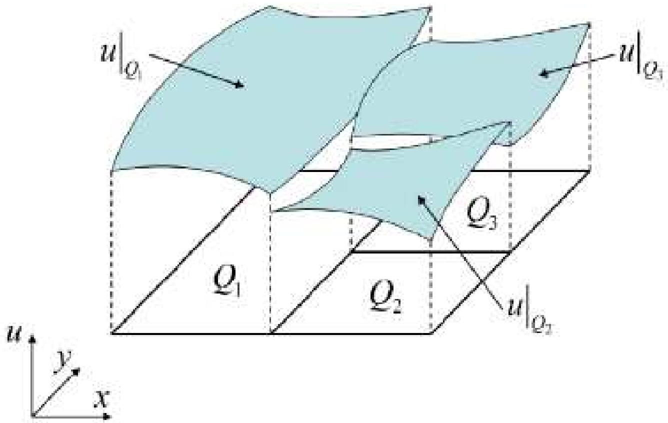


Source: Gassner [1]

	Complex geometries	High-order accuracy & hp-adaptivity	Explicit semi-discrete form	Conservation laws
FD	×	✓	✓	✓
FV	✓	×	✓	✓
FE	✓	✓	×	(✓)
DG	✓	✓	✓	✓

Source: Hesthaven & Warburton [5]

Discontinuous Galerkin (DG) spectral element methods may be a viable alternative



Source: Gassner [1]

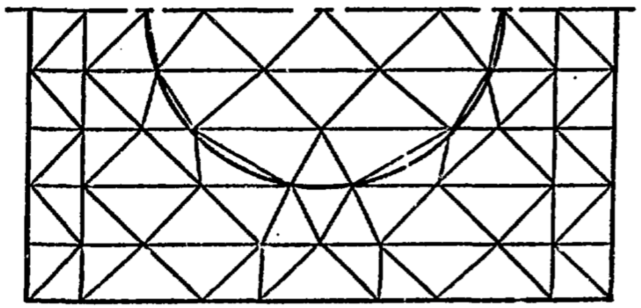
Basic characteristics of the scheme:

- Subdivision of domain into elements
- Polynomials are used to approximate solution inside elements
- Global solution is discontinuous
- Weak coupling of elements
- Non-conforming mesh topology possible
- Applicable to hyperbolic/parabolic problems

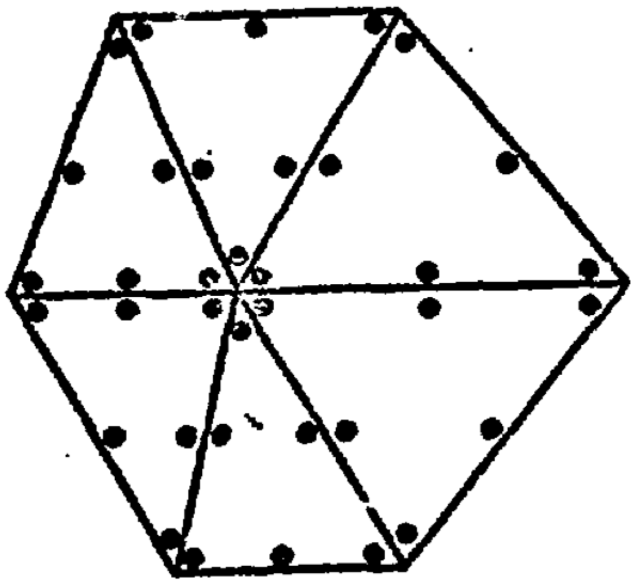
Conceptual similarities to

- FE methods: variational formulation
- FV methods: discontinuous on element faces

A brief history of discontinuous Galerkin (DG) methods



150 triangle mesh



Source: Reed and Hill [6]

- First proposed by Reed/Hill (1973) to solve steady-state neutron transport equations
- Extension to non-linear equations by Chavent and Salzano (1982)
- System of equations in 1D first presented by Cockburn, Lin et al. (1989), for multivariate problems by Cockburn and Shu (1998)
- Second-order derivatives formulation developed by Bassi and Rebay (1997)
- First use for acoustics by Atkins and Shu (1998)
- Since 2000 surge in application to variety of hyperbolic and hyperbolic/parabolic systems

Weak formulation is basis for DG approach

- Example: scalar hyperbolic conservation law in d dimensions

$$u_t + \nabla \cdot \underline{f}(u) = 0 \tag{1}$$

- Divide domain Ω into elements (cells) $Q \subset \Omega$, multiply with with test function $\phi(\underline{x})$, and integrate over element

$$\int_Q (u_t + \nabla \cdot \underline{f})\phi \,\mathrm{d}\underline{x} = 0 \tag{2}$$

- Use integration by parts to obtain weak formulation

$$\int_Q u_t \phi \,\mathrm{d}\underline{x} + \oint_{\partial Q} (\underline{f} \cdot \underline{n})\phi \,\mathrm{d}s - \int_Q \underline{f} \cdot \nabla\phi \,\mathrm{d}\underline{x} = 0 \tag{3}$$

Use polynomial ansatz inside the elements and numerical flux on boundary

- Insert polynomial ansatz function

$$u|_Q \approx u_Q = \sum_{j=1}^{N(p,d,Q)} \underbrace{a_j^Q(t)}_{\text{DOF}} \; \underbrace{\phi_j^Q(\underline{x})}_{\text{basis functions}} \tag{4}$$

- Choose basis functions as test functions (Galerkin approach), i.e. $\phi \in \{\phi_j^Q\}_{i=1}^N$
- Use <u>numerical flux</u> in surface integral: $\underline{f} \cdot \underline{n} \approx g(u^+, u^-)$
- Finally obtain mathematical formulation of DG method in weak form

$$\boxed{\underbrace{\int_Q u_t^Q \phi_i^Q \,\mathrm{d}\underline{x}}_{\text{time integral}} + \underbrace{\oint_{\partial Q} g(u^+, u^-)\phi_i^Q \,\mathrm{d}s}_{\text{surface integral}} - \underbrace{\int_Q \underline{f}(u^Q) \cdot \nabla \phi_i^Q \,\mathrm{d}\underline{x}}_{\text{volume integral}} = 0, \qquad i = 1, \ldots, N} \tag{5}$$

Numerical integration & Runge-Kutta scheme complete the method

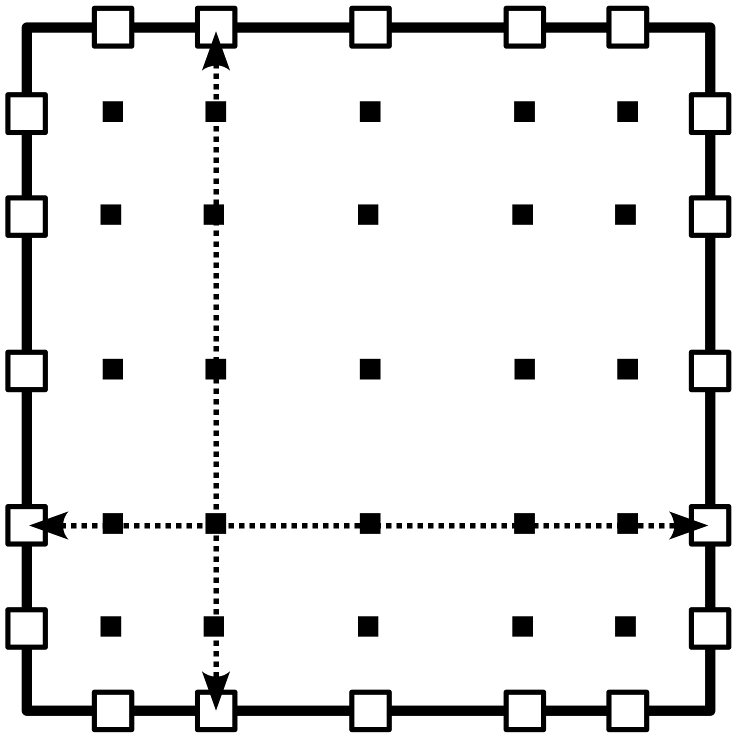
- Typical choices for polynomial representation and quadrature method:
  - polynomials: Legendre or Chebyshev
  - quadrature method: Gauss or Gauss-Lobatto

  (for hexahedral/square elements: efficient tensor product formulation possible)
- Transformation to reference element $Q_r \in [-1, 1]^d$
- After application of quadrature method and inversion of (element-local) mass matrix, one can reformulate the DG method as

$$\frac{\partial u_j^Q(t)}{\partial t} = R_j^Q(u_j^Q, t), \qquad j = 1, ..., N \tag{6}$$

- Now use regular ODE integration method, e.g. low-storage or low-dispersion Runge-Kutta scheme, to get solution at time $t' = t + \Delta t$

Applicability/further extension of the DG method



Source: Altmann, Beck et al. [8]

- Any hyperbolic/parabolic equation in conservative form can be discretized
- Systems of equations: $\underline{u}$ instead of u, use suitable numerical flux
- Convection-diffusion equations: reformulate to 1^{st} order system (c.f. Bassi and Rebay [7])
- Hybrid methods (reduce DOF on boundaries)
- Explicit filtering: use modal polynomial representation (i.e. 1, x, x^2, ...)
- Curved boundaries

Investigation of the DG method with underresolved turbulence

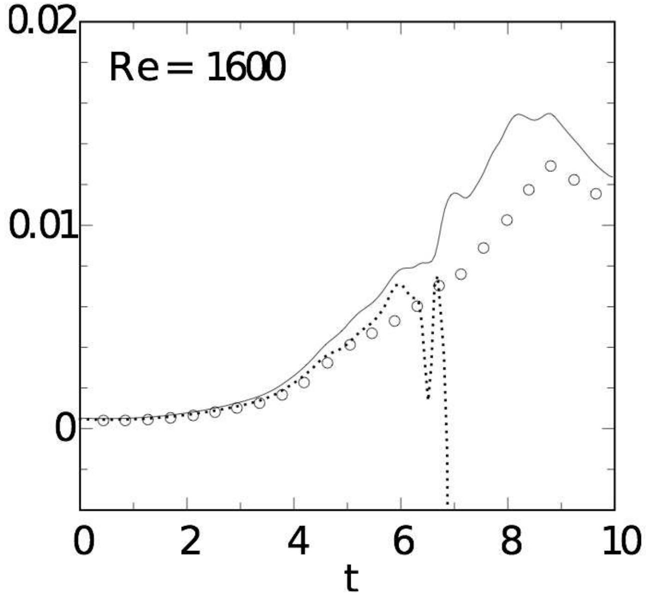


Figure: Kinetic energy dissipation rates at $\mathrm{Re} = 1600$; KES (—), Navier-Stokes (· · ·), DNS (○)

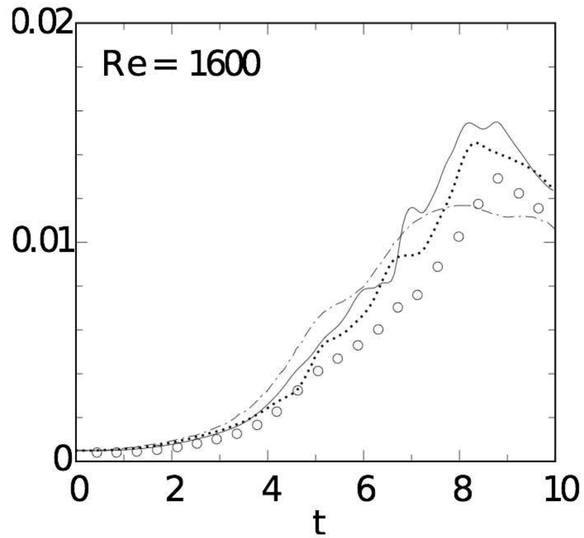
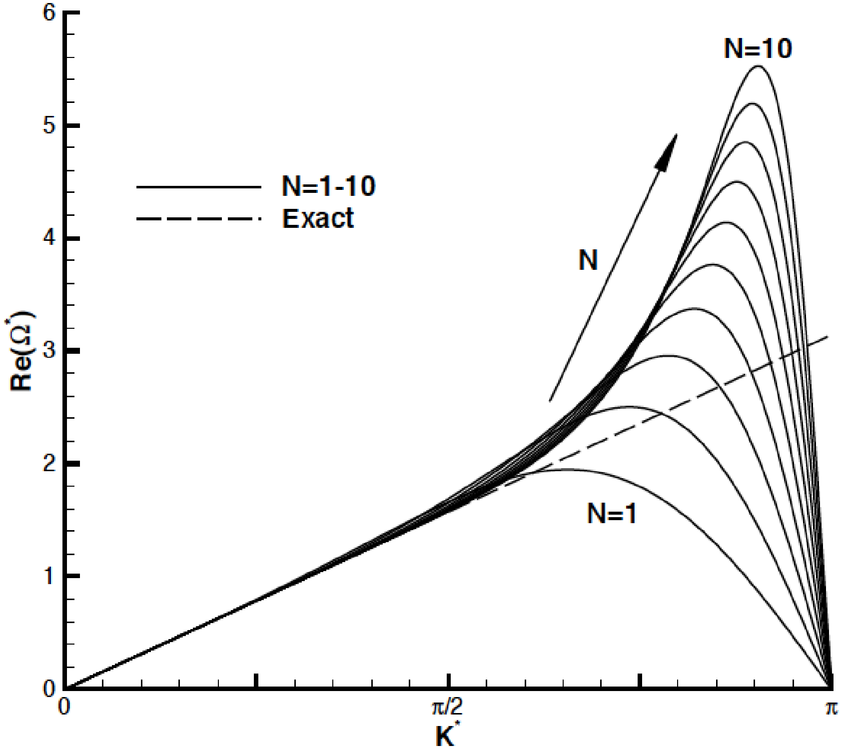
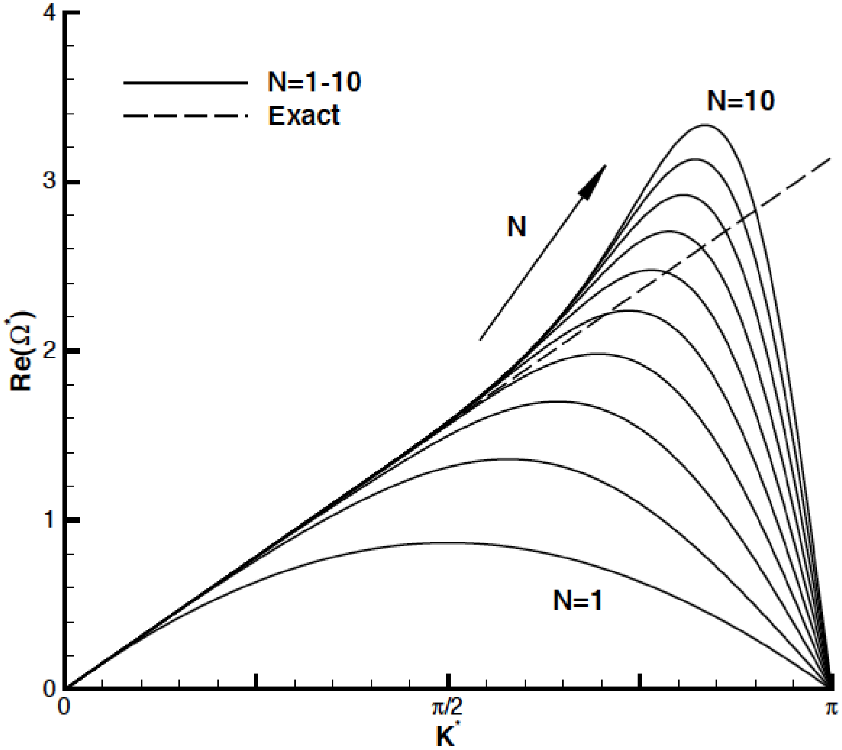


Figure: Kinetic energy dissipation rates at $\mathrm{Re} = 1600$; KES (—), ALDM (· · ·), dynamic Smagorinsky (− ·), DNS (○)

Analysis of dispersion relation for Gauss/Gauss-Lobatto DGSEM

Plots for Legendre-Gauss (left) and Legendre-Gauss-Lobatto (right)



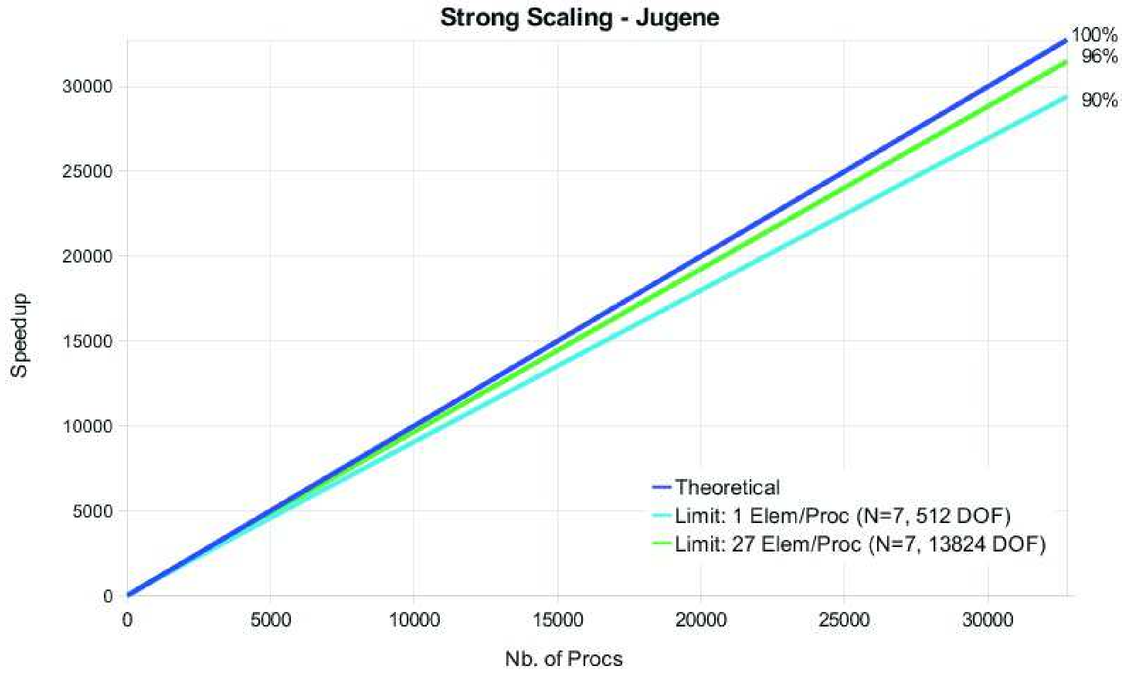


Points per wavelength for dispersion error $||\mathrm{Re}(\Omega^*) - K|| = 0.0001$

points \ *N*	1	2	3	4	5	6	7	8	9	10
Gauss	25.62	15.25	11.35	9.38	8.22	7.43	6.87	6.45	6.13	5.86
Gauss-Lobatto	153.69	34.45	18.50	13.06	10.41	8.88	7.87	7.16	6.64	6.22

Source: Altmann, Beck et al. [8]

Scaling results show good parallelization properties



Method	spec. CPU time (Nehalem) [μs]
cFD (O6)	4
Modal DG (N=5)	10
DGSEM (G, N=5)	2
DGSEM (GL, N=5)	1.6

Source: Altmann, Beck et al. [8]

Continued interest in further developing existing DG schemes

Excerpt of interesting current research projects:

- Munz, Gassner (Unversität Stuttgart): *underresolved turbulence, stabilization mechanisms, space-time extension*
- Qin, Krivodonova (University of Waterloo): *DG solutions on Cartesian grids with embedded geometries (cut cells)*
- Dahmen, May, Schütz (RWTH Aachen): *error estimation, hybrid mixed methods*
- Piperno, Duruflé (INRIA): *aeroacoustics, dissipation free methods, high order efficiency*

Conclusion: DG scheme has many desirable properties

Summary of the DG spectral element method

- high order formulation
- fast & efficient formulations available (tensor product structure)
- very compact scheme $\rightarrow$ highly parallelizable
- possibility to use existing methods for limiting, fluxes etc.
- supports non-conforming meshes $\rightarrow$ hp-refinement possible
- easy handling of complex geometries

$\Rightarrow$ **Promising features, but more research is necessary!**

References

[1] G. Gassner: *Discontinuous-Galerkin-Verfahren*, Institute of Aerodynamics and Gas Dynamics, Universität Stuttgart, 2013.

[2] F. Hindenlang: *Rechengitter: Klassifizierung, Anwendung*, Institute of Aerodynamics and Gas Dynamics, Universität Stuttgart, 2013.

[3] Y. Reymen, W. De Roeck, G. Rubio, M. Baelmans and W. Desmet: *A 3D discontinuous Galerkin method for aeroacoustic propagation*, ICSV12 - Twelth International Congress on Sound and Vibration, 2005.

[4] M. Bernacki and S. Piperno: *A dissipation-free time-domain discontinuous Galerkin method applied to three-dimensional linearized Euler equations around a steady-state non-uniform inviscid flow*, Journal of Computational Acoustics (14), 445-467, 2006.

[5] J. S. Hesthaven and T. Warburton: *Nodal Discontinuous Galerkin Methods*, Springer, 2008.

[6] W. H. Reed and T. R. Hill: *Triangular mesh methods for the neutron transport equation*, Los Alamos Scientific Laboratory Report LA-UR-73–479 19, 1973.

[7] F. Bassi and S. Rebay: *A high-order accurate discontinuous Galerkin finite element method for the numerical solution of the compressible Navier-Stokes equations*, Journal of Computational Physics (131), 267-279, 1997.

[8] C. Altmann, A. Beck, S. Fechter, G. Gassner, F. Hindenlang, F. Lörcher, M. Staudenmaier, C.-D. Munz: *An Overview of the Discontinuous Galerkin Developments at the Institute of Aerodynamics and Gasdynamics: Part II*, Institute of Aerodynamics and Gas Dynamics, Universität Stuttgart, 2011.

Image sources

- Noise footprint
  http://www.nasa.gov/topics/aeronautics/features/aircraft_noise_prt.htm
- Flight paths
  http://ryanmwithrow.files.wordpress.com/2009/01/typical-jet-west-east-flight-patterns1.gif
- A321 take-off
  http://www.airlinereporter.com/2011/12/airbus-delivers-their-7000th-aircraft-to-us-airways/
- Aircraft meshing
  http://www.uwyo.edu/mechanical/faculty-staff/dimitri-mavriplis/

Backup

Properties of the numerical flux, and possible choices

The used flux formulation $g(u^+, u^-)$ should be...

- consistent (i.e. $g(v, v) = f(v)$
- only depend on local values

Typical choices for flux formulations include

- local Lax-Friedrichs flux, i.e.

$$g(u^+, u^-) = \frac{1}{2}[(\underline{f}^+ \cdot \underline{n}^+) + (\underline{f}^- \cdot \underline{n}^-) - \alpha(u^+ - u^-)], \qquad \alpha = \max_{u \in [u^+, u^-]} |\underline{f}'(u)| \tag{7}$$

- Roe flux
- HLLC flux
- other (upwinding) flux formulations

back

Time derivative for a scalar, 2D, hyperbolic equation

$$\begin{aligned}\frac{\partial u_{ij}}{\partial t} =& \frac{1}{J_{ij}}\left(\tilde{f}(1,\eta_j)\frac{\psi_i(1)}{\omega_i} - \tilde{f}(-1,\eta_j)\frac{\psi_i(-1)}{\omega_i} - \sum_{n=1}^{p+1}\tilde{f}_{nj}\frac{\omega_n D_{ni}}{\omega_i}\right)\\ & \frac{1}{J_{ij}}\left(\tilde{g}(\xi_i,1)\frac{\psi_j(1)}{\omega_j} - \tilde{g}(\xi_i,-1)\frac{\psi_j(-1)}{\omega_j} - \sum_{l=1}^{p+1}\tilde{g}_{il}\frac{\omega_l D_{lj}}{\omega_j}\right)\end{aligned} \tag{8}$$

back

Summary of the DG spectral element method

- (almost) arbitrarily high order formulation
- high order $\rightarrow$ fewer grid points necessary $\rightarrow$ lower memory requirements
- fast & efficient formulations available (tensor product structure)
- very compact scheme $\rightarrow$ highly parallelizable
- easy formulation of boundary conditions (weak or strong)
- possibility to use existing methods for limiting, fluxes etc.
- supports non-conforming meshes $\rightarrow$ hp-refinement possible
- easy handling of complex geometries (structured/unstructured, hexahedron/tetrahedron, curved boundaries)

Disadvantages of discontinuous Galerkin methods:

- high order can become unstable (limiting/filtering necessary)
- increase of DOF on faces
- less efficient than FE methods for elliptic problems
- less mature than other methods (FE, FD, FV)